ON THE GEOMETRY OF BIFURCATION CURRENTS FOR QUADRATIC RATIONAL MAPS.

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ABSTRACT. We describe the behaviour at infinity of the bifurcation current in the moduli space of quadratic rational maps. To this purpose, we extend it to some closed, positive $(1,1)$-current on a two-dimensional complex projective space and then compute the Lelong numbers and the self-intersection of the extended current.

1. Introduction.

For any holomorphic family $(f_\lambda)_{\lambda\in M}$ of degree d rational maps on $\mathbb{P}^1$, the bifurcation locus is the subset of the parameter space M where the Julia sets J_λ of f_λ does not move continuously with λ. In their seminal paper [MSS], Mañé, Sad and Sullivan have shown that the bifurcation locus is nowhere dense in M and coincides with the closure of the set of parameters for which f_λ admits a non-persistent neutral cycle.
For the quadratic polynomial family $\left(z^2+\lambda\right)_{\lambda\in\mathbb{C}}$, the bifurcation locus is the boundary of the Mandelbrot set and, in particular, is bounded. The situation is much more complicated for quadratic rational maps. Their moduli space $\mathcal{M}_2$ can be identified with the complement in a complex projective space $\mathbb{P}^2$ of a line at infinity $\mathbb{L}_\infty$ around which the behaviour of the bifurcation locus is far to be completely understood.
The first results in this direction are due to J. Milnor [M] who studied the curves

$$\mathrm{Per}_n(w) := \{[f]\in\mathcal{M}_2 \text{ s.t. } f \text{ has a } n\text{-cycle of multiplier } w\}$$

in the projective space $\mathbb{P}^2$ (see Proposition 2.3). Sharper results have then been obtained by A. Epstein [E] and yields a precise description of the intersections of $\mathrm{Per}_n(w)$ with $\mathbb{L}_\infty$ (see Proposition 4.1). It should be stressed that the deformations of the Mandelbrot set, provided by intersecting the bifurcation locus with the *lines* $Per_1(t)$ for $|t|<1$ (see Proposition 2.4), play a central role in these investigations. In particular, the work of C. Petersen [P] on the collapsing of limbs (see the beginning of the fourth section) is crucial in Epstein's proof.

Currents not only provide an appropriate framework for studying bifurcations from a measure-theoretic viewpoint, but are also very well suited to investigate the asymptotic distribution of the hypersurfaces $\mathrm{Per}_n(w)$. Let us recall that L. DeMarco has proved that the bifurcation locus of any holomorphic family supports a closed, positive $(1,1)$-current [DeM1]. This current is denoted T_{bif} and called *bifurcation current*. We refer the reader to the survey [Du] or the lecture notes [Be] for a report on recent results involving bifurcation currents and further references.
The link between the bifurcation current T_{bif} and the hypersurfaces $\mathrm{Per}_n(w)$ relies on the fact that the Lyapunov exponent $L(\lambda)$ of f_λ (with respect to its maximal entropy measure) is a global potential for T_{bif} (see [DeM2] or [BB1]). G. Bassanelli and the first author have

indeed deduced from this property the convergence of the weighted integration currents on $\mathrm{Per}_n(w)$

$$\lim_n \tfrac{1}{d^n}[\mathrm{Per}_n(w)] = T_{\mathrm{bif}}$$

for $|w| < 1$ or for any $w \in \mathbb{C}$ when the hyperbolic parameters are dense in M (see [BB2]).

The present paper is devoted to the study of the behaviour at infinity of the bifurcation current in the moduli space $\mathcal{M}_2$ of quadratic rational maps. We first show that this current extends naturally to a closed, positive $(1,1)$-current $\hat{T}_{\mathrm{bif}}$ on $\mathbb{P}^2$. We relate this current with the hypersurfaces $\mathrm{Per}_n(w)$ and precise its support (see Theorem 3.1). We then use Epstein's result to compute the Lelong numbers of $\hat{T}_{\mathrm{bif}}$ at any point of the line at infinity $\mathbb{L}_\infty$ and get a precise description of the measure $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ (see Theorem 4.2). We finally describe the support of the so-called bifurcation measure $T_{\mathrm{bif}} \wedge T_{\mathrm{bif}}$ and compare this measure with $\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}$ (see Propositions 5.1 and 5.2).

Notations. The complex plane will be denoted $\mathbb{C}$ and the euclidean unit disc in $\mathbb{C}$ will be denoted $\mathbb{D}$. The k-dimensional complex projective space is denoted $\mathbb{P}^k$. A ball of radius r centered at x in some metric space is denoted $B(x,r)$.

2. Preliminaries.

2.1. Lelong numbers and currents in $\mathbb{P}^2$.

Let T be a closed, positive, $(1,1)$-current in $\mathbb{P}^2$. The *mass* of T on an open set $U \subset \mathbb{P}^2$ is given by

$$\|T\|_U := \int_U T \wedge \omega$$

where ω is the Fubini-Study form on $\mathbb{P}^2$. When V is an algebraic curve in $\mathbb{P}^2$ and $[V]$ is the integration current on V, then $\|[V]\|_{\mathbb{P}^2} = \deg(V)$.

For any $x \in \mathbb{P}^2$, the *Lelong number* of T at x is given by

$$\nu(T,x) := \lim_{r\to 0} \frac{1}{r^2}\|T\|_{B(x,r)}.$$

These numbers somehow measure the singularities of T. If V is an algebraic curve in $\mathbb{P}^2$, then $\nu([V],x)$ is the multiplicity of V at x. We will mainly use the two following properties of Lelong numbers (see [Dem] Proposition 5.12 page 160 and Corollary 7.9 page 169).

Theorem 2.1 (Demailly). (1) *If $T_n \longrightarrow T$, then for any $x \in \mathbb{P}^2$,*

$$\nu(T,x) \geq \limsup_{n\to+\infty} \nu(T_n,x).$$

(2) *If T_1 and T_2 are such that $T_1 \wedge T_2$ is well-defined, then for any $x \in \mathbb{P}^2$,*

$$\nu(T_1 \wedge T_2, x) \geq \nu(T_1,x)\cdot\nu(T_2,x).$$

2.2. The moduli space $\mathcal{M}_2$ of quadratic rational maps.

The space Rat_2 of quadratic rational maps on $\mathbb{P}^1$ may be viewed as a Zarisky-open subset of $\mathbb{P}^5$ on which the group of Möbius transformations acts by conjugation. The *moduli space* $\mathcal{M}_2$ is, by definition, the quotient resulting from this action. Denote by α, β, γ the multipliers of the three fixed points of $f \in \mathrm{Rat}_2$ and $\sigma_1 = \alpha+\beta+\gamma$, $\sigma_2 = \alpha\beta+\alpha\gamma+\beta\gamma$ and

$\sigma_3 = \alpha\beta\gamma$ the symmetric functions of these multipliers. Milnor has proved that (σ_1, σ_2) is a good parametrization of $\mathcal{M}_2$ (see [M]).

Theorem 2.2 (Milnor). *The map $[f] \in \mathcal{M}_2 \longmapsto (\sigma_1, \sigma_2) \in \mathbb{C}^2$ is a canonical biholomorphism.*

To study the curves

$$\mathrm{Per}_n(w) := \{[f] \in \mathcal{M}_2 \text{ s.t. } f \text{ has a } n\text{-cycle of multiplier } w\},$$

it is useful to compactify $\mathcal{M}_2$. In this context, it turns out that the projective compactification

$$[f] \in \mathcal{M}_2 \hookrightarrow [\sigma_1 : \sigma_2 : 1] \in \mathbb{P}^2$$

is suitable. Denote by $\mathbb{L}_\infty$ the line at infinity of $\mathcal{M}_2$, i.e.

$$\mathbb{L}_\infty = \{[\sigma_1 : \sigma_2 : 0] \in \mathbb{P}^2 \ / \ (\sigma_1, \sigma_2) \in \mathbb{C}^2 \setminus \{0\}\}.$$

Milnor has described the behavior of $\mathrm{Per}_n(w)$ at infinity as follows (see [M] Lemmas 3.4 and section 4).

Proposition 2.3 (Milnor).
1. *For any $w \in \mathbb{C}$ the curve $\mathrm{Per}_1(w)$ is a line in $\mathbb{P}^2$ whose equation in $\mathbb{C}^2$ is $(w^2+1)\lambda_1 - w\lambda_2 - (w^3+2) = 0$ and which intersects the line at infinity $\mathbb{L}_\infty$ at the point $[w : w^2+1 : 0]$.*
2. *For any $n \geq 2$ and any $w \in \mathbb{C}$, the curve $\mathrm{Per}_n(w)$ is an algebraic curve in $\mathbb{P}^2$ whose degree equals the number $d(n) \sim 2^{n-1}$ of n-hyperbolic components of the Mandelbrot set. In addition, the intersection $\mathrm{Per}_n(w) \cap \mathbb{L}_\infty$ is contained in the set $\{[1 : u + 1/u : 0] \in \mathbb{P}^2 \ / \ u^q = 1 \text{ with } q \leq n\}$.*

For reasons which will appear to be clear later, we shall denote by $\mathbb{L}_{\infty,\mathrm{bif}}$ the subset of $\mathbb{L}_\infty$ defined by

$$\mathbb{L}_{\infty,\mathrm{bif}} := \{[1 : e^{i\theta} + 1/e^{i\theta} : 0] \in \mathbb{P}^2 / \theta \in [0, 2\pi]\}.$$

Golberg and Keen showed how the Mandelbrot set $\mathbf{M}$ determines the connectedness locus for quadratic rational maps having an attracting fixed point (see [GK] Section 1 and [BB2] Theorem 5.4 for a potential theoretic approach).

Theorem 2.4 (Goldberg-Keen, Bassanelli-Berteloot). *There exists a holomorphic motion $\sigma : \mathbb{D} \times \mathrm{Per}_1(0) \longrightarrow \mathbb{C}^2$ such that $\mathbf{M}_t := \sigma_t(\mathbf{M}) \Subset \mathrm{Per}_1(t) \cap \mathbb{C}^2$ is the connectedness locus of the line $\mathrm{Per}_1(t)$. Moreover, if $|w| \leq 1$ and $n \geq 1$, then $\mathrm{Per}_1(t) \cap \mathrm{Per}_n(w) \subset \mathbf{M}_t$.*

2.3. Bifurcation currents in $\mathcal{M}_2$.

Every rational map f of degree $d \geq 2$ on the Riemann sphere admits a maximal entropy measure μ_f. The Lyapunov exponent of f with respect to this measure is defined by

$$L(f) = \int_{\mathbb{P}^1} \log |f'| \mu_f.$$

It turns out that the function $L : \mathrm{Rat}_2 \longrightarrow L(f)$ is *p.s.h* and continuous on Rat_2. We still denote by L the function induced on $\mathcal{M}_2$. The *bifurcation current* T_{bif} is a closed positive (1, 1)-current on $\mathcal{M}_2$ which may be defined by

$$T_{\mathrm{bif}} = dd^c L.$$

As it has been shown by DeMarco [DeM2], the support of T_{bif} concides with the bifurcation locus of the family in the classical sense of Mañe-Sad-Sullivan (see also [BB1], Theorem 5.2). Using properties of the Lyapounov exponent, Bassanelli and the first author proved that the curves $\mathrm{Per}_n(0)$ equidistribute the bifurcation current (see [BB2]).

Theorem 2.5 (Bassanelli-Berteloot). *The sequence* $2^{-n}[\mathrm{Per}_n(0)]$ *converges to* T_{bif} *in the sense of currents in* $\mathcal{M}_2$. *Moreover,* $2^{-n}[\mathrm{Per}_n(0)]|_{\mathrm{Per}_1(0)}$ *converges weakly to* $\Delta(L|_{\mathrm{Per}_1(0)}) = \frac{1}{2}\mu_{\mathbf{M}}$, *where* $\mu_{\mathbf{M}}$ *is the harmonic measure of* $\mathbf{M}$.

As the function L is continuous, one can define the *bifurcation measure* μ_{bif} of the moduli space $\mathcal{M}_2$ as the Monge-Ampère mass of the function L, i.e.

$$\mu_{\mathrm{bif}} := (dd^cL)^2 := dd^c\,(Ldd^cL).$$

This measure has been introduced by Bassanelli and the first author [BB1]. Buff and Epstein also studied it in [BE]. Recall that a rational map f is said to be strictly poscritically finite if each critical point of f is preperiodic but not periodic. We denote by $\mathcal{SPCF}$ the set of classes of quadratic strictly postcritically finite rational maps. We also denote by $\mathcal{S}$ the set of *Shishikura* maps, i.e. $\mathcal{S} := \{[f_0] \in \mathcal{M}_2 \ / \ f_0$ has 2 distinct neutral cycles$\}$. Combining the work of Bassanelli and the first author with that of Buff and Epstein, we have the following.

Theorem 2.6 (Bassanelli-Berteloot, Buff-Epstein).

$$\mathrm{supp}(\mu_{\mathrm{bif}}) = \overline{\mathcal{SPCF}} = \overline{\mathcal{S}}.$$

These results are still valid in moduli spaces of degree d rational maps for any $d \geq 2$. Notice that Buff and the second author [BG] have proved that flexible Lattès maps belong to $\mathrm{supp}(\mu_{\mathrm{bif}})$.

3. Extension of the bifurcation current to Milnor's compactification.

As the current T_{bif} is a weak limit of weighted integration currents on curves which are actually defined on $\mathbb{P}^2$, one may expect to naturally extend it to $\mathbb{P}^2$. We will show how this is possible and prove some basic properties of the extended current. More precisely, we establish the following result which may be considered as a measure-theoretic counterpart of Milnor's description of the bifurcation locus in $\mathcal{M}_2$.

Theorem 3.1. *There exists a closed positive* $(1,1)$*-current* $\hat{T}_{\mathrm{bif}}$ *on* $\mathbb{P}^2$ *whose mass equals* $1/2$ *and such that:*

1. $\hat{T}_{\mathrm{bif}}|_{\mathbb{C}^2} = T_{\mathrm{bif}}$,
2. $2^{-n}[\mathrm{Per}_n(0)]$ *converges to* $\hat{T}_{\mathrm{bif}}$ *in the sense of currents on* $\mathbb{P}^2$,
3. $\mathrm{supp}(\hat{T}_{\mathrm{bif}}) = \mathrm{supp}(T_{\mathrm{bif}}) \cup \mathbb{L}_{\infty,\mathrm{bif}}$.

We shall use the two following lemmas which are of independant interest. The first one is essentially due to Milnor (see Theorem 4.2 in [M] or Theorem 2.3 in [BB2]). The second one will be proved at the end of the section.

Lemma 3.2. $\|[\mathrm{Per}_n(0)]\|_{\mathbb{P}^2} \sim 2^{n-1}$.

Lemma 3.3. $\overline{\bigcup_{n\geq 1} \mathrm{Per}_n(0)} \cap \mathbb{L}_\infty = \mathbb{L}_{\infty,\mathrm{bif}}$.

Proof. We first justify the existence of $\hat{T}_{\mathrm{bif}}$. According to the Skoda-El Mir Theorem (see [Dem] Theorem 2.3 page 139), the trivial extension of T_{bif} through the line at infinity $\mathbb{L}_\infty$ is a closed positive $(1,1)$-current on $\mathbb{P}^2$ if T_{bif} has locally bounded mass near $\mathbb{L}_\infty$. The Lemma 3.2, combined with the fact that $2^{-n}[\mathrm{Per}_n(0)]$ converges to T_{bif} on $\mathbb{C}^2$ (see

Theorem 2.5), immediatly yields $\|T_{\text{bif}}\|_{\mathbb{C}^2} \leq 1/2$. The extension $\hat{T}_{\text{bif}}$ of T_{bif} therefore exists and $\|\hat{T}_{\text{bif}}\|_{\mathbb{P}^2} \leq 1/2$.

Let us now prove that $2^{-n}[\text{Per}_n(0)]$ converges to $\hat{T}_{\text{bif}}$ on $\mathbb{P}^2$ and $\|\hat{T}_{\text{bif}}\|_{\mathbb{P}^2} = 1/2$. By Lemma 3.2, $(2^{-n}[\text{Per}_n(0)])_{n\geq 1}$ is a sequence of closed positive $(1,1)$-currents with uniformly bounded mass on $\mathbb{P}^2$. According to the compactness porperties of such families of currents, it suffices to show that $\hat{T}_{\text{bif}}$ is the only limit value of $2^{-n}[\text{Per}_n(0)]$.

Assume that $2^{-n_k}[\text{Per}_{n_k}(0)]$ converges to T on $\mathbb{P}^2$. By Siu's Theorem (see [Dem] Theorem 8.16 page 181), one has $T = S + \alpha[\mathbb{L}_\infty]$, where S has no mass on $\mathbb{L}_\infty$. But, by Lemma 3.3, the line $\mathbb{L}_\infty$ is not contained in $\text{supp}(T)$. Thus $\alpha = 0$ and T has no mass on $\mathbb{L}_\infty$. Since $T|_{\mathbb{C}^2} = \lim_k 2^{-n_k}[\text{Per}_{n_k}(0)] = T_{\text{bif}}$, this shows that T is actually the trivial extension of T_{bif} through $\mathbb{L}_\infty$ and therefore, according to the first part of the proof, equals $\hat{T}_{\text{bif}}$.

Now, Lemma 3.2 immediatly yields $\|\hat{T}_{\text{bif}}\|_{\mathbb{P}^2} = 1/2$. □

Proof of Lemma 3.3. Let $\sigma : \text{Per}_1(0) \times \mathbb{D} \longrightarrow \bigcup_{|u|<1} \text{Per}_1(u)$ be the holomorphic motion given by Theorem 2.4. Let us set $\mathbf{M}_u := \sigma_u(\mathbf{M})$. As $\mathbf{M}$ is compact and σ is continuous one has

$$\bigcup_{|u|\leq r<1} \mathbf{M}_u \Subset \mathbb{C}^2 \text{ for all } 0 < r < 1.$$

Let us also recall that $\text{Per}_n(0) \cap \text{Per}_1(u) \subset \mathbf{M}_u$ for $n \geq 2$ and $u \in \mathbb{D}$.

We now proceed by contradiction and assume that there exists

$$\zeta \in \left(\overline{\bigcup_n \text{Per}_n(0)} \cap \mathbb{L}_\infty\right) \setminus \mathbb{L}_{\infty,\text{bif}}.$$

By Proposition 2.3, $\zeta \in \text{Per}_1(u_0)$ for some $u_0 \in \mathbb{D}$. Let us pick $\lambda_k \in \text{Per}_{n_k}(0)$ such that $\lambda_k \to \zeta$. Then there exists $u_k \in \mathbb{D}$ such that $u_k \to u_0$ and $\lambda_k \in \text{Per}_1(u_k)$, which is impossible since

$$\lambda_k \in \text{Per}_{n_k}(0) \cap \text{Per}_1(u_k) \subset \mathbf{M}_{u_k} \subset \bigcup_{|u|<r} \mathbf{M}_u$$

for some $|u_0| < r < 1$. □

4. **Lelong numbers of the bifurcation current at infinity.**

The aim of the present section is to compute the Lelong numbers of $\hat{T}_{\text{bif}}$ at any point. This is related to previous works of Petersen ([P]) and Epstein ([E]) which we briefly describe.

Let $\heartsuit$ be the main cardioid of the Mandelbrot set and $\mathcal{L}_{p/q}$ the *p/q-limb* of $\mathbf{M}$ (see [Br] page 84). Denote by $d_{p/q}(n)$ the number of n-hyperbolic component of $\mathcal{L}_{p/q}$ and set

$$D_{p/q}(n) = \begin{cases} d_{p/q}(n) & \text{if } p/q = 1/2, \\ 2d_{p/q}(n) & \text{otherwise.} \end{cases}$$

Let σ be the holomorphic motion of $\text{Per}_1(0)$ given by Theorem 2.4 and

$$\infty_{p/q} := [1 : 2\cos(2\pi p/q) : 0]$$

if $1 \leq p \leq q/2$ and $p \wedge q = 1$. Petersen has proved that the limb $\sigma_u(\mathcal{L}_{p/q})$ of $\mathbf{M}_u$ disappears when u tends non-tangentially to $e^{-2i\pi p/q}$. Using this result, Epstein has precisely described the intersection $\text{Per}_n(w) \cap \mathbb{L}_\infty$. He namely proved the following:

Proposition 4.1 (Epstein)**.** *For any $w \in \mathbb{C}$ and any $n \geq 2$,*

$$[\mathrm{Per}_n(w)] \wedge [\mathbb{L}_\infty] = \sum_{\substack{1\leq p\leq q/2\leq n/2\\ p\wedge q=1}} \nu([\mathrm{Per}_n(w)], \infty_{p/q})\delta_{\infty_{p/q}} = \sum_{\substack{1\leq p\leq q/2\leq n/2\\ p\wedge q=1}} D_{p/q}(n)\delta_{\infty_{p/q}}.$$

From the above Proposition and by using the previous section, we deduce the following.

Theorem 4.2. (1) *The Lelong numbers of* $\hat{T}_{\mathrm{bif}}$ *are given by*

$$\nu(\hat{T}_{\mathrm{bif}}, a) = \begin{cases} 1/6 & \text{if } a = \infty_{1/2}, \\ 1/(2^q-1) & \text{if } a = \infty_{p/q} \text{ and } , q \geq 3 \\ 0 & \text{if } a \in \mathbb{P}^2 \setminus \{\infty_{p/q} \ / \ p\wedge q = 1, \ 1 \leq p \leq q/2\}. \end{cases}$$

(2) *The measure* $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ *is discrete and given by*

$$\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] = \sum_{\substack{1\leq p\leq q/2\\ p\wedge q=1}} \nu(\hat{T}_{\mathrm{bif}}, \infty_{p/q})\delta_{\infty_{p/q}}.$$

The proof requires the two following lemmas. The first one is a consequence of Theorem 3.1 and the second one relies on a simple computation. They will be proved at the end of the section.

Lemma 4.3. *The measure* $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ *is a well-defined positive measure on* $\mathbb{P}^2$ *of mass* 1/2.

Lemma 4.4. *Let* $\mu := \dfrac{1}{6}\delta_{\infty_{1/2}} + \displaystyle\sum_{\substack{1\leq p\leq q/2\\ p\wedge q=1, \ q\geq 3}} \frac{1}{2^q-1}\delta_{\infty_{p/q}}$*, then* μ *has mass* 1/2.

Proof. First observe that $\nu(\hat{T}_{\mathrm{bif}}, a) = 0$ when $a \notin \mathbb{L}_{\infty,\mathrm{bif}}$, since then $\hat{T}_{\mathrm{bif}}$ has a continuous potential in a neighborhood of a. Let us now pick $1 \leq p \leq q/2$ such that $p \wedge q = 1$. By item (3) of Theorem 3.1 and Theorem 2.1, we have

$$\nu(\hat{T}_{\mathrm{bif}}, \infty_{p/q}) \geq \limsup_{n\to\infty} \nu(2^{-n}[\mathrm{Per}_n(0)], \infty_{p/q}).$$

Proposition 4.1 states that $\nu([\mathrm{Per}_n(0)], \infty_{p/q}) = D_{p/q}(n)$ is the number of n-hyperbolic components of the union $\mathcal{L}_{p/q} \cup \mathcal{L}_{-p/q}$ of limbs of the Mandelbrot set. Thus, by Theorem 2.5,

$$2^{-n}\nu([\mathrm{Per}_n(0)], \infty_{p/q}) \longrightarrow \tfrac{1}{2}\mu_{\mathbf{M}}(\mathcal{L}_{p/q} \cup \mathcal{L}_{-p/q}).$$

On the other hand, Bullett and Sentenac have proved that $\mu_{\mathbf{M}}(\mathcal{L}_{p/q}) = \mu_{\mathbf{M}}(\mathcal{L}_{-p/q}) = \frac{1}{2^q-1}$ (see [BS]). This gives $\nu(\hat{T}_{\mathrm{bif}}, \infty_{p/q}) \geq \frac{1}{2^q-1}$ if $p/q \neq 1/2$ and $\nu(\hat{T}_{\mathrm{bif}}, \infty_{1/2}) \geq \frac{1}{2(2^2-1)} = \frac{1}{6}$.

Let $a \in \mathbb{L}_\infty$. By Theorem 2.1 , we have $\nu(\hat{T}_{\mathrm{bif}}\wedge[\mathbb{L}_\infty], a) \geq \nu(\hat{T}_{\mathrm{bif}}, a)\nu([\mathbb{L}_\infty], a)$. As $\mathbb{L}_\infty$ is a line, we get $\nu(\hat{T}_{\mathrm{bif}}\wedge[\mathbb{L}_\infty], a) \geq \nu(\hat{T}_{\mathrm{bif}}, a)$. For $a = \infty_{p/q}$, we thus have $\nu(\hat{T}_{\mathrm{bif}}\wedge[\mathbb{L}_\infty], \infty_{p/q}) \geq \frac{1}{2^q-1}$ if $q \geq 3$ and $\nu(\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty], \infty_{1/2}) \geq \frac{1}{6}$, which we restate as

$$\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] \geq \mu.$$

Since, according to Lemmas 4.3 and 4.4, the positive measures $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ and μ have the same mass, this yields $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] = \mu$. We thus get point (2) and $\nu(\hat{T}_{\mathrm{bif}}, \infty_{p/q}) = \frac{1}{2^q-1}$ for $q \geq 3$ and $\nu(\hat{T}_{\mathrm{bif}}, \infty_{1/2}) = \frac{1}{6}$.

From $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] \geq \nu(\hat{T}_{\mathrm{bif}}, a)\delta_a$ for any $a \in \mathbb{L}_\infty$ and $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] = \mu$ we get $\nu(\hat{T}_{\mathrm{bif}}, a) = 0$ for $a \neq \infty_{p/q}$. □

Proof of Lemma 4.3. Let us first remark that, by Theorem 3.1, $\mathrm{supp}(\hat{T}_{\mathrm{bif}})\cap\mathbb{L}_\infty = \mathbb{L}_{\infty,\mathrm{bif}}$ and let us decompose $\mathbb{P}^2$ as the disjoint union of the line $\mathrm{Per}_1(0)$ and (a copy of) $\mathbb{C}^2$. Let us stress that with these notations one has $\mathbb{L}_\infty\setminus\mathrm{Per}_1(0)\cap\mathbb{L}_\infty\subset\mathbb{C}^2$. By Proposition 2.3, the set $\mathbb{L}_{\infty,\mathrm{bif}}$ is compact in $\mathbb{C}^2$. By definition, the current $\hat{T}_{\mathrm{bif}}$ has a potential u in $\mathbb{C}^2$ which is continuous in $\mathbb{C}^2\setminus\mathbb{L}_\infty$. By item (3) of Theorem 3.1, this potential is actually continuous in $\mathbb{C}^2\setminus\mathbb{L}_{\infty,\mathrm{bif}}$. Let B_1 be a ball of $\mathbb{C}^2$ containing $\mathbb{L}_{\infty,\mathrm{bif}}$ and $(B_i)_{i\geq 2}$ be a covering of $\mathbb{C}^2\setminus B_1$ by balls such that $\overline{B_i}\cap\mathbb{L}_{\infty,\mathrm{bif}}=\emptyset$ for all $i\geq 2$.

As $\{u \text{ is unbounded}\}\subset\mathbb{L}_{\infty,\mathrm{bif}}$ and $\mathbb{L}_{\infty,\mathrm{bif}}\cap\partial B_i=\emptyset$ and B_i is pseudoconvex for any $i\geq 1$, a result of Demailly asserts that $\hat{T}_{\mathrm{bif}}|_{\mathbb{C}^2}\wedge[\mathbb{L}_\infty]=dd^c u\wedge[\mathbb{L}_\infty]$ is well-defined (see [Dem] Proposition 4.1 page 150). Since, by Proposition 2.3 and Theorem 4.2, $\mathbb{L}_\infty$ and $\mathrm{supp}(\hat{T}_{\mathrm{bif}})$ don't intersect in a neighborhood of $\mathrm{Per}_1(0)$, the measure $\hat{T}_{\mathrm{bif}}\wedge[\mathbb{L}_\infty]$ is a well-defined positive measure on $\mathbb{P}^2$.

Finally, Bézout Theorem asserts that $\|\hat{T}_{\mathrm{bif}}\wedge[\mathbb{L}_\infty]\|_{\mathbb{P}^2}=\|\hat{T}_{\mathrm{bif}}\|_{\mathbb{P}^2}\cdot\|[\mathbb{L}_\infty]\|_{\mathbb{P}^2}=1/2$. □

Proof of Lemma 4.4. Denote by $\phi(n)$ the Euler function. As the sets $\{p \text{ s.t. } 1\leq p\leq q/2,\ p\wedge q=1\}$ and $\{q-p \text{ s.t. } 1\leq p\leq q/2,\ p\wedge q=1\}$ have same cardinality, we get

$$\begin{aligned}\mu(\mathbb{P}^2) &= \frac{1}{6}+\sum_{\substack{1\leq p\leq q/2,\\ p\wedge q=1,\ q\geq 3}}\frac{1}{2^q-1}=\frac{1}{2(2^2-1)}+\frac{1}{2}\sum_{\substack{1\leq p<q,\\ p\wedge q=1,\ q\geq 3}}\frac{1}{2^q-1}\\ &= \frac{1}{2}\sum_{q\geq 2}\left(\sum_{\substack{1\leq p<q,\\ p\wedge q=1}}\frac{1}{2^q-1}\right)=\frac{1}{2}\sum_{q\geq 2}\frac{\phi(q)}{2^q-1}.\end{aligned}$$

A classical result (see [HW] Theorem 309 page 258) asserts that the series $\sum_{n\geq 1}\phi(n)\frac{x^n}{1-x^n}$ locally uniformly converges on $\mathbb{D}$ and that its sum is $\frac{x}{(1-x)^2}$. Therefore,

$$\mu(\mathbb{P}^2)=\frac{1}{2}\left(\sum_{q\geq 1}\frac{\phi(q)}{2^q-1}-\phi(1)\right)=\frac{1}{2}\left(\sum_{q\geq 1}\frac{\phi(q)(\frac{1}{2})^q}{1-(\frac{1}{2})^q}-\phi(1)\right)=\frac{1}{2}.$$

□

5. Behavior of the bifurcation measure near infinity.

One often compares the moduli space $\mathcal{M}_2$ of quadratic rational maps with the moduli space $\mathcal{P}_3$ of cubic polynomials. In this section, we enlight some important differences between these two spaces. We first show that the bifurcation measure is not compactly supported in $\mathcal{M}_2$.

Proposition 5.1. *The cluster set of the support of μ_{bif} in $\mathbb{P}^2$ is precisely $\mathbb{L}_{\infty,\mathrm{bif}}$.*

Proof. By Theorem 3.1, it suffices to show that $\mathbb{L}_{\infty,\mathrm{bif}}$ is accumulated by points of $\mathrm{supp}(\mu_{\mathrm{bif}})$. Recall that $\mathrm{Per}_1(e^{2i\pi\nu})\cap\mathbb{L}_\infty=\{[1:2\cos(2\pi\nu):0]\}$ for any $0\leq\nu\leq 1$ (see Proposition 2.3). Let us fix $0<\theta_0<1$. For $\theta>0$ small enough, the lines $\mathrm{Per}_1(e^{2i\pi\theta_0})$ and $\mathrm{Per}_1(e^{2i\pi(\theta-\theta_0)})$ do not intersect on $\mathbb{L}_\infty$ and therefore,

$$\{\lambda(\theta)\}:=\mathrm{Per}_1(e^{2i\pi\theta_0})\cap\mathrm{Per}_1(e^{2i\pi(\theta-\theta_0)})\subset\mathbb{C}^2.$$

Since $\lambda(\theta)$ has two distinct neutral fixed points, it belongs to $\mathrm{supp}(\mu_{\mathrm{bif}})$ (see Theorem 2.6). It remains to check that $\lim_{\theta\to 0}\lambda(\theta) = [1 : 2\cos(2\pi\theta_0) : 0]$. By the holomorphic index formula (see [M]), the multiplier $\gamma(\theta)$ of the third fixed point of $\lambda(\theta)$ is given by

$$\gamma(\theta) = \frac{2 - (e^{2i\pi\theta_0} + e^{2i\pi(\theta-\theta_0)})}{1 - e^{2i\pi\theta}}.$$

As $\lim_{\theta\to 0}|\gamma(\theta)| = +\infty$, one has $\lim_{\theta\to 0}\|\lambda(\theta)\| = +\infty$. The conclusion follows, since $\lambda(\theta)$ stays on $\mathrm{Per}_1(e^{2i\pi\theta_0})$ and $\mathrm{Per}_1(e^{2i\pi\theta_0}) \cap \mathbb{L}_\infty = \{[1 : 2\cos(2\pi\theta_0) : 0]\}$. □

We would like to extend μ_{bif} as a reasonable bifurcation measure on $\mathbb{P}^2$. To this aim, we compare μ_{bif} with $(\hat{T}_{\mathrm{bif}})^2$. We prove the following.

Proposition 5.2. *There exists a positive measure μ_∞ supported by $\mathbb{L}_{\infty,\mathrm{bif}}$ such that*

$$\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}} = \mu_{\mathrm{bif}} + \frac{1}{36}\delta_{\infty_{1/2}} + \sum_{\substack{1\le p\le q/2\\ p\wedge q=1, q\ge 3}} \frac{1}{(2^q-1)^2}\delta_{\infty_{p/q}} + \mu_\infty.$$

Les us stress that our previous results ensure the existence of μ_∞ as a non-negative measure. A recent result due to Kiwi and Rees concerning the number of (n,m)-hyperbolic components of $\mathcal{M}_2$ allows to see that μ_∞ is actually a positive measure.

Proof. Arguing exactly as in the proof of Lemma 4.3, one sees that $\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}$ is a well-defined positive measure on $\mathbb{P}^2$. By definition, $\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}$ and μ_{bif} coincide on $\mathbb{C}^2$. To prove the existence of μ_∞, it thus only remains to justify that

$$\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}} \ge \mu_{\mathrm{bif}} + \frac{1}{36}\delta_{\infty_{1/2}} + \sum_{\substack{1\le p\le q/2\\ p\wedge q=1, q\ge 3}} \frac{1}{(2^q-1)^2}\delta_{\infty_{p/q}}.$$

By Theorem 2.1 and Theorem 4.2, we have

$$\nu(\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}, \infty_{p/q}) \ge \nu(\hat{T}_{\mathrm{bif}}, \infty_{p/q})^2 = \frac{1}{(2^q-1)^2}, \text{ when } q \ge 3$$

and $\nu(\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}, \infty_{1/2}) \ge 1/36$. The existence of μ_∞ then follows, since

$$\left(\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}\right)\big|_{\mathbb{L}_\infty} \ge \sum_{\substack{1\le p\le q/2\\ p\wedge q=1}} \nu(\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}, \infty_{p/q})\delta_{\infty_{p/q}}.$$

Let us now show that $\mu_\infty > 0$. One easily deduce from the convergence of $2^{-n}[\mathrm{Per}_n(0)]$ towards T_{bif} in any family (see [BB2]) that

$$\mu_{\mathrm{bif}} = \lim_m \lim_n 2^{-(n+m)}[\mathrm{Per}_n(0) \cap \mathrm{Per}_m(0)]$$

on $\mathbb{C}^2$. Thus $\mu_{\mathrm{bif}}(\mathbb{C}^2) \le \limsup_m \limsup_n 2^{-(n+m)}[\mathrm{Per}_n(0) \cap \mathrm{Per}_m(0)](\mathbb{C}^2)$. Kiwi and Rees proved recently that the number of (n,m)-hyperbolic components in $\mathbb{C}^2$ is at most

$$\left(\frac{5}{48}2^n - \frac{1}{8}\sum_{q=2}^{n} \frac{\phi(q)\nu_q(n)}{2^q-1}\right)2^m + \mathcal{O}(2^m),$$

where $\nu_q(n) \sim 2^{n-1}/(2^q-1)$ and $\nu_q(n) \ge 2^{n-1}/(2^q-1) - 1/2$ (see [KR] Theorem 1.1). A standard transversality statement asserts that this number actually coincides with $[\mathrm{Per}_n(0) \cap \mathrm{Per}_m(0)](\mathbb{C}^2)$ (see [BB2] Theorem 5.2). Thus

$$\mu_{\mathrm{bif}}(\mathbb{C}^2) \leq \frac{5}{48} - \frac{1}{16}\sum_{q\geq 2}\frac{\phi(q)}{(2^q-1)^2}.$$

Let us now proceed by contradiction, assuming that $\mu_\infty = 0$. Since $\hat{T}_{\mathrm{bif}}$ has mass $1/2$, Bézout Theorem gives $\|\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}\|_{\mathbb{P}^2} = 1/4$. Therefore,

$$\frac{1}{4} = \|\mu_{\mathrm{bif}}\| + \frac{1}{36} + \sum_{\substack{1\leq p\leq q/2 \\ p\wedge q=1, q\geq 3}} \frac{1}{(2^q-1)^2} = \|\mu_{\mathrm{bif}}\| + \frac{1}{2}\sum_{q\geq 2}\frac{\phi(q)}{(2^q-1)^2} - \frac{1}{36},$$

which yields $\frac{25}{56} \leq \sum_{q\geq 2}\frac{\phi(q)}{(2^q-1)^2}$. We then have

$$\frac{25}{56} \leq \sum_{q\geq 2}\frac{\phi(q)}{(2^q-1)^2} \leq \frac{\phi(2)}{6} + \frac{\phi(3)}{28} + \frac{1}{8}\sum_{q\geq 4}\frac{\phi(q)}{2^q-1}$$

which is impossible, since $\sum_{q\geq 1}\frac{\phi(q)}{2^q-1} = 2$ (see proof of Lemma 3.2), $\phi(2) = 1$ and $\phi(3) = 2$. □

Remark 5.1. *To underline the contrast with the moduli space of cubic polynomials $\mathcal{P}_3$, we recall that the bifurcation measure is compactly supported and coincides with $\hat{T}^2_{\mathrm{bif}}$ in $\mathcal{P}_3$.*

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